On the Metric Dimension of Joins of Two Graphs

Shahida.A.T, M.S.Sunitha

1,2 – -Department of Mathematics, National Institute of Technology Calicut,
NIT Campus-673601,India , Fax:+91 495 2287250,
Ph: 9846793843.
1-shahisajid@gmail.com
2-sunitha@nitc.ac.in

Abstract—For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ **of vertices and a vertex** *v* **in a connected graph** *G*, **the (metric) representation of** *v* **with respect to** *W* **is the** *k*-**vector** *r(v/W)* = (d(v, w₁), d(v, w₂), …, d(v, wₖ)), **where** *d(x, y)* **represents the distance between the vertices x and y. A resolving set of minimum cardinality is called a minimum resolving set or a basis and the cardinality of a basis for G is its dimension dim** *G*. **For the graph** $G_1 = (V_1, E_1)$ **and** $G_2 = (V_2, E_2)$ **their join is denoted by** $G_1 + G_2$ **is the graph whose vertex set is** $V_1 \cup V_2$ **and the edge set is** $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. **In this paper, we determine the metric dimension of join of paths, paths and cycles, path and stars, complete graphs, complete graphs and paths.**

Keywords—**Join, Metric dimension, Metric basis, Resolving set.**

——————————◆——————————

1. INTRODUCTION

The metric dimension problem was first introduced in 1975 by Slater [9], and independently by Harary and Melter [7] in 1976. This parameter has been studied for $K_n, P_n, k_{m,n}, C_n, P$ (Petersen graph), grids, trees, multi-dimensional grids, Torus networks, graph operations etc. Let G be a connected graph of order $n \geq 2$, then dim(G) = n-1 if and only if G = Kₙ and for $n \geq 4$, dim (G) = n – 2 if and only if $G = K_{r,s}(r, s \geq 1), G = K_r + \overline{k}_s(r \geq 1, s \geq 2)$ or $G = K_r + (K_1 \cup K_s)(r, s \geq 1)$ [9]. And dim(G) = 1 if and only if G = Pₙ[8]. Some bounds for metric dimension of join of graphs G and H as dim (G) + dim (H) ≤ dim (G + H) and max (dim (G), dim (H)) ≤ min {dim (G) + $|H|$, dim (H) + $|G|$}-1 are established in [3]. The concept of (minimum) resolving set has proved to be useful and/or related to a variety of fields. The concept of minimum metric dimension has applications in the field of robotics[10]. A robot is a mechanical device which is made to move in space with obstructions around. It has neither the concept of direction nor that of visibility. But it is assumed that it can sense the distances to a set of landmarks. A basic problem in chemistry to provide mathematical representations for a set of chemical compounds in a way that gives distinct representation to distinct compounds. As described in [8], the structure of a chemical compound can be represented by a labelled graph whose vertex and edge labels specify the atom and bond types, respectively. Other applications of resolving sets arise in various areas including coin weighing problem, drug discovery[8], robot navigation [11], network discovery and verification [13], connected joins in graphs[1] and strategies for the

mastermind game[12]. For a survey of results in metric dimension, we refer to Chartrand and Ping[5].

2. PRELIMINARIES

A graph G = (V, E) is an ordered pair consisting of a nonempty set V of elements called vertices and a set E of unordered pairs of vertices called edges. A graph is said to be connected if there is a path between every pair of vertices. If the graph is connected, the distance between every two vertices a and b, denoted by d(a, b), is the length of a shortest path joining a and b. A graph having no loops or multiple edges is called a simple graph [6]. All the graphs considered in this paper are undirected, simple, finite and connected. The order and size of G are denoted by n(or m) and k respectively. We use standard terminology, the terms not defined here may found in [2], [4] and [7].

The idea of resolving sets has appeared in the literature previously [7]. A vertex $x \in V(G)$ is said to resolve a pair of vertices $\{u, v\}$ in G if $d(u, x) \neq d(v, x)$. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of V(G) and for any vertex $v \in V$, the (metric) representation of v with respect to W is the k-vector which is denoted and defined as $r(v/W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ and if $r(v/W) = r(u/W)$ then u = v for all $u, v \in V(G)$. A resolving set of minimum cardinality is called a minimum resolving set. A resolving set W of G is defined to be connected if the sub graph induced by W is a nontrivial connected sub graph of G. A resolving set W of G is said to be independent if no two vertices in W are adjacent. A minimum resolving set is usually called a basis for G. A resolving set W is called a minimal resolving set of a connected graph G if no proper subset of W is a resolving set of G. The minimum cardinality of a minimal resolving set of G is called the metric dimension of G and is denoted by dimG. The metric dimension of almost all graphs depends upon the number of vertices in the graph. But for some graphs like paths, cycles, their metric dimensions are not depending on the number of vertices in the graphs.

3 METRIC DIMENSION OF JOIN OF PATHS AND PATHS WITH OTHER FAMILIES OF GRAPHS

In this section we study about the resolving sets of join of paths, paths and cycles and paths and stars. The following theorem provides the metric dimension of join of a path to itself.

Theorem.3.1. For every positive integer n, the metric dimension of $P_n + P_n$ is

$$\dim(P_n + P_n) = \begin{cases} 1 & \text{if } n = 1 \\ 3 & \text{if } 2 \leq n \leq 3 \\ n & \text{if } n \geq 4 \end{cases} \text{(3.1)}$$

Proof: For $n = 1$, $P_1 + P_1 = P_2$. Therefore $\dim(P_1 + P_1) = 1$. For $n = 2, P_2 + P_2 = K_4$. Therefore $\dim(P_2 + P_2) = 3$. For $n \geq 3$ consider $G = P_n + P_n$. Vertices of G is labelled as in Figure-1.
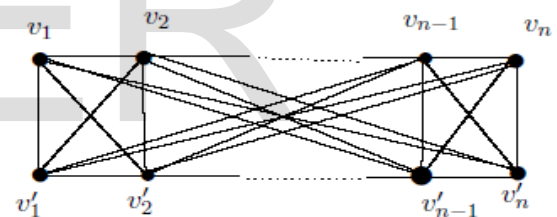

Figure-1

Here distance between any two vertices is less than or equal to 2. And also $d(v_i, v_j') = 1 \forall i, j = 1, 2, \ldots, n.$ For $n = 3, W = \{v_1, v_2, v_3\}$ form a basis for $P_3 + P_3$.

Claim: For $n \geq 4, \dim(G) \geq n$.

If we assume $\dim(G) < n$, then we can find out at least one pair of vertices in $V \backslash W$ such that they have the same metric representation with respect to W. Therefore $\dim(G) \geq n.$

From calculations we get if n is an odd number then $\{v_1', v_{3,...}', v_n', v_2', v_4', v_6', \cdots, v_{n-1}'\}$ is a basis and if n is even then $\{v_1', v_{3,...}', v_{n-1}', v_2', v_4', \cdots, v_n'\}$ form a basis for G. Therefore dim(G)=n.

Theorem. 3.2. For given positive integers m, n the metric dimension of $(P_n + P_m)$ is

$$\dim(P_n + P_m) = \begin{cases} 1 & \text{if } m = n = 1 \\ 2 & \text{if } 2 \le m \le 3 \\ \left\lfloor \dfrac{m}{2} \right\rfloor + n - 1 & \text{if } n \ge 1, m \ge 4 \end{cases}$$

(3.2)

Proof: Let $G = P_n + P_m$. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and $u_1, u_2, \ldots, u_m$ be the vertices of P_m.

Case.1: $n = m = 1$

Result follows by Theorem. 3.1.

Case.2: If $n = 1, m = 2$ then $(P_1 + P_2) \cong K_3$.

Therefore $\dim(P_1 + P_2) = 2$.

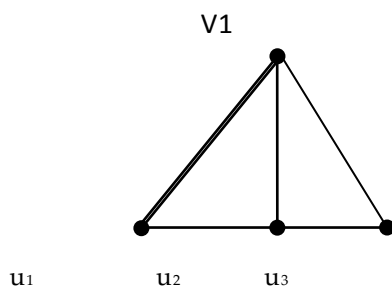If $n = 1, m = 3$ then vertices of $P_1 + P_3$ are labelled as in Fig-2.

V1

u₁   u₂   u₃

Figure 2

Then $W = \{u_1, u_2\}$ form a basis for G. Therefore $\dim(P_1 + P_3) = 2$.

Case. 3: if $n \ge 1$ and $m \ge 4$

For $n = 1$ and $m \ge 4$

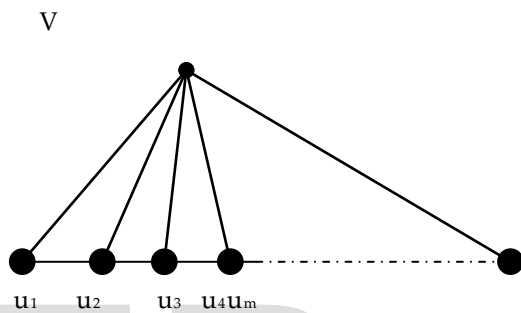Let $G = P_1 + P_m$ and $v, u_1, u_2, \ldots, u_m$ be the vertices of G as shown in Fig-3.

V

u₁   u₂   u₃ u₄uₘ

Figure 3

Then $W = \left\{ u_1, u_{m-1}, u_3, u_{m-3}, \cdots, u_{\left\lfloor \frac{m}{2} \right\rfloor} \right\}$ form a basis for G. Therefore

$$\dim(P_1 + P_m) = \left\lfloor \frac{m}{2} \right\rfloor, m \ge 4.$$

Similarly $\dim(P_2 + P_m) = \left\lfloor \dfrac{m}{2} \right\rfloor + 1, m \ge 4$.

Since

$$W = \left\{ v_1, u_1, u_{m-1}, u_3, u_{m-3}, \cdots, u_{\left\lfloor \frac{m}{2} \right\rfloor} \right\}$$

form a basis for $P_2 + P_m$, where $v1$ is a vertex of P_2.

More generally for $n \ge 1$ and $m \ge 4$

$$\dim(P_n + P_m) = \left\lfloor \frac{m}{2} \right\rfloor + n - 1, m \ge 4.$$

Remark.3.3.The subgroup induced by the vertices of a basis for the join of two paths are connected or equivalently their does not

exist an independent basis for join of two paths.

Theorem.3.4. The metric dimension of join of two graphs is 2 if and only if it is of the form $P_1 + P_m$ where $2 \leq m \leq 5$.

Proof: Let G bethe join of two graphs with $\dim(G) = 2$. The only graphs with dimension 2 and which do not have independent basis are K_3, C_4 and $P_1 + P_m$ where $2 \leq m \leq 5$. And by remark.3.3 there does not exist an independent basis for join of two graphs. Since $K_3 = (P_1 + P_2)$ is the join of P_1 and $P_2 \Rightarrow G = (P_1 + P_m), m = 2,3,4,5$ are the only graphs which have dimension 2.

Converse part is follows by Theorem. 3.2.

Theorem.3.5. For given positive integer m and n = 1,

$$\dim(P_1 + C_m) = \begin{cases} 3 & \text{if } 3 \leq m \leq 9 \\ 4 & \text{if } 10 \leq m \leq 11 \\ 5 & \text{if } 12 \leq m \leq 13 \\ \dots \\ \left\lfloor \left(\frac{k}{2}\right) \right\rfloor + 1 & \text{if } k-1 \leq m \leq k \end{cases}$$

Theorem.3.6. For given positive integer m and n = 2,

$$\dim(P_2 + C_m) = \begin{cases} 4 & \text{if } m = 3 \\ \left\lceil \dfrac{m-1}{2} \right\rceil + 1 & \text{if } m \geq 4 \end{cases}$$

(3.4)

Theorem.3.7. For a given positive integer m and k=0, 1, 2, …

$$\dim(P_3 + C_m) = \begin{cases} 4 & \text{if } m = 3 \\ 4 + k & \text{if } 4 + 2k \leq m \leq 5 + 2k \end{cases}$$

(3.5)

Theorem.3.8. For a given positive integer n t
$\dim(P_1 + K_{1,n}) = n.$

Proof: Let $G = P_1 + K_{1,n}$, be the join of P_1 and $K_{1,n}$, where $V(P_1) = \{v\}$ and $V(K_{1,n}) = \{u_1, u_2, \dots, u_{n+1}\}$

Claim: W = (v, u₁, u₂…uₙ₋₁) is a basis for G.

Every element in V \ W get a unique metric representation with respect to W. Therefore W is a resolving set for G. If we remove any vertex from W then it is not a resolving set for G. Let W′ be any subset of V (G)with (n-1) vertices including v, then among thevertices of V\W we can find out atleast onepair of vertices having the same metricrepresentation with respect to W′. Therefore W′ is not a resolving set for G, then dim (P₁+K₁, ₙ)=n.

Theorem.3.9. For a given positive integer n,

$$\dim(P_2 + K_{1,n}) = \begin{cases} 3 & \text{if } n = 1 \\ n+1 & \text{if } n \geq 2 \end{cases}.$$

(3.6)

Open Problem. 3.10. For given positive integers m and n find the metric dimension of Pₘ + K₁,ₙ.

4 METRIC DIMENSION OF JOIN OF TWO COMPLETE GRAPHS

Theorem. 4.1. The metric dimension of join of K₁ and Kₙ is $\dim(K_1 + K_n) = n$.

Proof: The join $K_1 + K_n = K_{n+1}$ and $\dim K_{n+1} = n$. Therefore $\dim K_1 + K_n = n$.

Theorem. 4.2. Given positive integers m and n,

$$\dim(K_m + K_n) = \dim(K_m) + \dim(K_n) + 1$$

Proof: The join $K_n + K_m = K_{n+m}$. Therefore
$$\dim(K_n + K_m) = n + m - 1 = (n-1) + (m-1) + 1 = \dim K_n + \dim K_m + 1$$
.

Now we study about the dimension of join of paths and complete graph as follows.

Theorem.4.3. For positive integers n, m. The metric dimension of $P_m + K_n$ is

$$\dim(P_m + K_n) = \dim P_m + \dim K_n + 1$$

Proof: Let and $v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n$ be the vertices of G. We have following cases.

Case. 1: $m \le 2, n \le 3$

Follows fromTheorem 4.2.

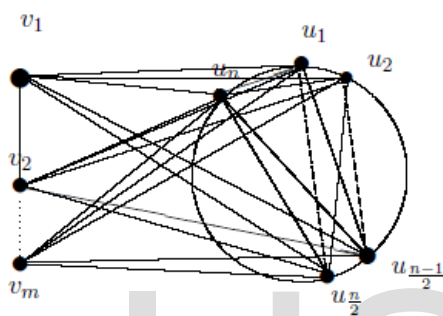**Case.2:** $m \ge 3, n \ge 3$, consider the following graph $G = P_m + K_n$ in   Figure-4

Figure-4

Claim: Every basis of G contains two adjacent vertices v_i, v_{i+1}, where $1 \le i \le m$.

Proof for claim: If possible assume that $k = \dim(G)$ and there exists a basis $W = \{v_i, u_1, u_2, \ldots, u_{k-1}\}$ for G such that $v_i \in W$ and $v_{i-1}, v_{i+1} \notin W$. Since each v_j is adjacent to every $u_k, k = 1, 2, \ldots, n$ there are 3 sub cases.

Sub Case.1: If $i = 1$ then $v_1 \in W$ and the metric representation of v_2 is (1, 1, ..., 1) with respect to W and also we can find atleast one vertex among u_j such that its metric representation with respect to W is (1, 1, ..., 1). Which contradicts the choice of W. Therefore $v_2 \in W$.

Sub Case.2: If $i = n$ then $v_n \in W$ and v_{n-1} and at least one vertex among u_j have the

same metric representations with respect to W. We arrive at a contradiction as in case. 1.

Sub Case.3: If $1 < i < n$ then $v_i \in W$ and v_{i-1} and v_i have the same metric representation (1, 1, ..., 1) with respect to W. Therefore either v_{i-1}, v_i or v_i, v_{i+1} in W.

hence the proof for the claim is complete.

For the case of K_n every n vertices are adjacent to each other and each of $v_i, i = 1, 2, \ldots, m$. Therefore we must choose n – 1 vertices from u_i, u_2, …,u_n to form a basic. Hence dim (G) = 2+n-1=1+(n-1) + 1 = $\dim P_m + \dim K_n + 1$

5. CONCLUSION

In this paper, we determine the metric dimension of join of paths, paths and cycles, path and stars, complete graphs, complete graphs and paths. It isalso characterised that $\dim(G_1 + G_2) = 2$ if and onlyif $G_1 = P1$ and $G_2 = P_m, 2 \le m \le 5$.

ACKNOWLEDGEMENT

REFERENCES

[1] A. Sebo and E. Tannier, On metric generators of Graphs, Math.Oper. Res., 29(2004), 383-393.

[2] Buckley F and Harary F, Distance in Graphs, Addision-Wesley, 1990.

[3] Carmen Hernando, Mer`ceMora, Ignacio M. Pelayo, Carlos Seara, On the metric dimension of some families of graphs, Electronic Notes in Discrete Mathematics 22(2005) 129-133.

[4] Gary Chartrand, Ping Zhang, Introduction to graph theory, Tata McGrew-Hill Publishing Company Limited, new Delhi, 2006.

[5] G. Chartrand, L. Eroh, M. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of graphs, Discrete Appl. Math., 105(2000), 99-113.

[6] F.Harary, Graph Theory, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1969.

[7] F. Harary, R.A. Melter, On the metric dimension of a graph, ArsCombin. 2 (1976), 191-195.

[8] G. Chartrand, L. Eroh, M. Johnson, O.R. Oellermann, Resolvabilityin graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000) ,99-113.

[9] P.J. Slater, Leaves of trees, Congr. Numer. 14 (1975), 549-559.

[10] S. Khuller, B.Ragavachari, and A. Rosenfeld, Localization in graphs, Technical Report CS-TR-3326,University of Maryland at College Park, 1994.

[11] V. Chvtal, Mastermind, Combinatorica, 3(1983), 325-329.

[12] Z. Beerliova, F. Eberhand, T. Erlebach, A.Hall, M.hoffman, M. Mihalak and L. Ram, Network discovery and verification, IEEE j. on selected areas in communications, 24(2006), 2168-2181.

.